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# Quantum Field Theory of Polarization of Light by a Polarizer

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We develop a quantum field theory analysis of the polarization of light by a polarizer by analyzing the photon Green function with the appropriate boundary conditions. Inherent in the study is a time evolutionary study of the system. The photon Green function is obtained as a photon moves from an emitter to various detectors set up relative to the polarizer. Upon using a completeness relation of the polarization vectors of light, the amplitudes of the relevant processes are extracted in a fully field-theoretic treatment in terms of photons.

## 1. INTRODUCTION

We develop a quantum field theory treatment of the practical problem of polarization of light by a polarizer. The photon Green function is set up and solved in detail with the appropriate boundary conditions. The latter is solved by tracing the time evolution of the system as a photon moves from an emitter to various detectors set up relative to a polarizer. Upon using a completeness relation with respect to polarization vectors, the relevant amplitudes for photon detection are extracted, from which the intensities for the various processes are computed. In this investigation, we were much inspired by the fascinating, but nontechnical, treatment of light given by Feynman (1985) and the abundant literature (e.g., Kennedy *et al.*, 1980; Deutsch and Candelas, 1979; Schwinger *et al.*, 1978; Balian and Duplantier, 1977; see also Manoukian, 1987a,b, 1992, 1993) dealing with the role of a quantum mechanical particle in a typical everyday situation. As in all these investigations, the boundary surface, that is, the polarizer itself, is replaced by appropriate boundary conditions, as also done in classical physics, rather

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than considering a quantum mechanical model for it as being made up of atoms and so on.

## 2. THE PHOTON GREEN FUNCTION

Upon using the definitions

$$E^i = F^{0i}, \qquad B^i = \tfrac{1}{2}\epsilon^{ijk}F^{jk}, \qquad \rho = \frac{J^0}{c} \tag{1}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2}$$

in Maxwell's equations

$$\nabla \cdot \mathbf{E} = \rho, \qquad \nabla \times \mathbf{B} = \partial_0 \mathbf{E} + \frac{\mathbf{J}}{c} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\partial_0 \mathbf{B} \tag{4}$$

we may write

$$\partial_\mu F^{\mu i} = -\frac{J^i}{c}, \qquad J^0 = -\frac{\partial_i J^i}{\partial_0} \tag{5}$$

We work in the celebrated temporal gauge $A^0 = 0$, to obtain from (5)

$$[(-\vec{\boldsymbol{\partial}}^2 + \partial^{02})\delta^{ij} + \partial^i\partial^j]A^j(x) = \frac{J^i(x)}{c} \tag{6}$$

Upon taking the vacuum-expectation value of equation (6) (with $J^i \neq 0$) and defining the photon Green function as the solution of

$$[(-\vec{\boldsymbol{\partial}}^2 + \partial^{02})\delta^{ik} + \partial^i\partial^k]D^{jk}(x', x) = \delta^{ij}\delta(x', x) \tag{7}$$

with appropriate boundary conditions to be spelled out, or of

$$[(-\vec{\boldsymbol{\partial}}'^2 + \partial^{0\prime 2})\delta^{ik} + \partial'^i\partial'^k]D^{kj}(x', x) = \delta^{ij}\delta(x', x) \tag{8}$$

we have

$$\langle A^i(x')\rangle = \frac{1}{c}\int (dx)\, D^{ij}(x', x)J^j(x)\langle 0_+ | 0_-\rangle \tag{9}$$

The vacuum-to-vacuum transition amplitude $\langle 0_+ | 0_-\rangle$ has the well-known form (see, e.g., Manoukian, 1986a)

$$\langle 0_+ | 0_- \rangle = \exp\left[\frac{i}{2\hbar c^3} \int (dx')\,(dx)\, J^i(x') D^{ij}(x', x) J^i(x)\right] \tag{10}$$

In particular, the expectation value of the electric field components $E^i(x')$ for $J^i \neq 0$ are given by

$$\langle E^i(x') \rangle = \frac{1}{c} \int (dx)\, \partial'^0 D^{ij}(x', x) J^j(x) \langle 0_+ | 0_- \rangle \tag{11}$$

The polarizer will be set up to constitute the $y$–$z$ plane, through the origin, such that it polarizes light in a direction perpendicular to the $y$ axis, that is, it "eats up" the $y$ component of the electric field at $x = 0$. Here $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x, y, z)$. Hence the appropriate boundary conditions are $\langle E^2(x') \rangle = 0$ at $x' = 0$ and $\langle E^a(x') \rangle$, $a = 1, 3$, are continuous at $x' = 0$. In terms of the photon Green function $D^{ij}(x', x)$ the boundary conditions are

$$D^{2j}(x', x)|_{x'=0} = 0 \tag{12}$$

$$D^{aj}(x', x)|_{x'=-0} = D^{aj}(x', x)|_{x'=+0} \tag{13}$$

$(x' = x'^1)$. The external current $J^i(x)$ will be written as (e.g., Schwinger, 1970; Manoukian, 1986b)

$$J^i(x) = J^i_1(x) + J^i_2(x) + J^i_3(x) \tag{14}$$

where the currents $J^i_2(x)$ and $J^i_3(x)$ are switched on after the current $J^i_1(x)$ is switched off. $J^i_1(x)$ will be identified with the emitter, and $J^i_2(x)$ and $J^i_3(x)$ with detectors. The supports in space of $J^i_1(x)$ and $J^i_2(x)$ will be taken to be in the region $x \equiv x^1 < 0$ and that of $J^i_3(x)$ in the region $x \equiv x^1 > 0$. The $J^i_3(x)$ detects a photon that would pass through the polarizer and is polarized, and $J^i_2(x)$ detects a photon which does not pass through the polarizer.

The various processes of interest will be obtained from (Manoukian, 1986b)

$$J^i_3(x') D^{ij}(x', x) J^j_1(x) \qquad \text{and} \qquad J^i_2(x') D^{ij}(x', x) J^j_1(x)$$

(see Section 4). Accordingly, the Green function will be eventually sandwiched between the currents in the above manner. This will allow us to fix $x$ to be in the region $x < 0$ in $D^{ij}(x', x)$, since the support in space of the current $J^i_1(x)$ lies in this region. We next consider the regions $x' < 0$ and $x' > 0$, respectively.

### 2.1. The $x' < 0$ Region

Here we are interested in $J_2^i(x')D^{ij}(x', x)J_1^j(x)$. For $x' < 0$, we may write

$$D^{ij}(x', x) = D_0^{ij}(x', x) + D_-^{ij}(x', x) \tag{15}$$

where $D_0^{ij}(x', x)$ is a particular solution of (7) and (8), and $D_-^{ij}(x', x)$ is a solution of the homogeneous equation corresponding to those of (7) and (8). That is,

$$[(-\boldsymbol{\partial}'^2 + (\partial'^0)^2)\delta^{ik} + \partial'^i\partial'^k]D_-^{kj}(x', x) = 0 \tag{16}$$

This in turn gives the equivalent equations

$$(-\boldsymbol{\partial}'^2 + (\partial'^0)^2)D_-^{ij}(x', x) = 0 \tag{17}$$

$$(-\boldsymbol{\partial}^2 + \partial^{02})D_-^{ij}(x', x) = 0 \tag{18}$$

$$\partial'^i D_-^{ij}(x', x) = 0, \qquad \partial^j D_-^{ij}(x', x) = 0 \tag{19}$$

We solve for the Green function $D^{ij}(x', x)$ for the causal relation $x'^0 > x^0$. The particular solution $D_0^{ij}(x', x)$ to (7) is well known:

$$D_0^{ij}(x', x) = i\int \frac{d^2\mathbf{K}}{(2\pi)^2}\int \frac{dq}{2\pi}\,\frac{e^{i\mathbf{K}\cdot(\mathbf{x}'_{11}-\mathbf{x}_{11})}}{2k} \times e^{iq(x'-x)}e^{-ik(x'^0-x^0)}\left(\delta^{ij} - \frac{k^ik^j}{k^2}\right) \tag{20}$$

where $x_{11} = (y, z)$, $\mathbf{k} = (q, \vec{\mathbf{K}})$, and $k = (q^2 + \mathbf{K}^2)^{1/2}$. To solve for $D_-^{ij}(x', x)$ we write

$$D_-^{ij}(x', x) = \int \frac{d^2\mathbf{K}}{(2\pi)^2}\int \frac{dk^0}{2\pi}\int \frac{dq'\,dq}{2\pi}\, e^{i\mathbf{K}\cdot(\mathbf{x}'_{11}-\mathbf{x}_{11})}e^{iq'x'}e^{-iqx} \times e^{-ik^0(x'^0-x^0)}D_-^{ij}(q', q, \mathbf{K}, k^0) \tag{21}$$

where (18) gives

$$(q^2 + \mathbf{K}^2 - k^{02})D_-^{ij}(q', q, \mathbf{K}) = 0 \tag{22}$$

With $x'^0 > x^0$, the emitter $J_1^i(x)$ emits a photon with a positive energy $k^0 > 0$ to be detected by $J_2^i(x')$. That is, (22) implies that we may write

$$D_-^{ij}(q', q, \mathbf{K}, k^0) = \frac{\pi}{k}\,\delta(k^0 - k)F_-^{ij}(q', q, \mathbf{K}) \tag{23}$$

On the other hand, (17) implies from (23) that

$$(q'^2 + \mathbf{K}^2 - (\mathbf{K}^2 + q^2))F_-^{ij}(q', q, \mathbf{K}) = 0 \tag{24}$$

Here the emitter $J_1^i(x)$ emits a photon with the first component $q$ of its momentum positive ($q > 0$) if it is to reach the polarizer. If the photon does not pass through the polarizer, then (after reflecting off) it is detected by $J_2^i(x')$, having the first component $q'$ of its momentum of opposite sign $q' < 0$. That is, from (24) *and* (22) we have

$$F_-^{ij}(q', q, \mathbf{K}) = i\delta(q' + q)G_-^{ij}(q, \mathbf{K}) \tag{25}$$

All told, this gives

$$D_-^{ij}(x', x) = i \int \frac{d^2\mathbf{K}}{(2\pi)^2} \int \frac{dq}{2\pi} \frac{e^{i\mathbf{K}\cdot(\mathbf{x}'_{\perp} - \mathbf{x}_{\perp})}}{2k} \times e^{-iq(x'+x)} e^{-ik(x'^0 - x^0)} G_-^{ij}(q, \mathbf{K}) \tag{26}$$

where $k = (q^2 + \mathbf{K}^2)^{1/2}$ as before.

## 2.2. The $x' > 0$ Region

Here we are interested in the expression for $J_3^i(x')D^{ij}(x', x)J_1^j(x)$. For $x' > 0$, the Green function $D^{ij}(x', x) = D_+^{ij}(x', x)$ satisfies the homogeneous differential equations corresponding to (7), (8). That is,

$$[(-\boldsymbol{\partial}'^2 + (\partial'^0)^2\delta^{ik} + \partial'^i\partial'^k]D_+^{kj}(x', x) = 0 \tag{27}$$

$$[(-\boldsymbol{\partial}^2 + \partial^{02})\delta^{ik} + \partial^i\partial^k]D_+^{jk}(x', x) = 0 \tag{28}$$

which lead to

$$(-\boldsymbol{\partial}'^2 + (\partial'^0)^2)D_+^{ij}(x', x) = 0 \tag{29}$$

$$(-\boldsymbol{\partial}^2 + \partial^{02})D_+^{ij}(x', x) = 0 \tag{30}$$

$$\partial'^i D_+^{ij}(x', x) = 0, \qquad \partial^j D_+^{ij}(x', x) = 0 \tag{31}$$

A similar analysis as for $D_-^{ij}(x', x)$, with the basic difference that $q'$ has to be positive ($q' > 0$) to reach the $J_3^i(x')$ detector, then gives

$$D^{ij}(x', x) = D_+^{ij}(x', x) = i \int \frac{d^2\mathbf{K}}{(2\pi)^2} \int \frac{dq}{2\pi} \frac{e^{i\mathbf{K}\cdot(\mathbf{x}'_{\perp} - \mathbf{x}_{\perp})}}{2k} \times e^{iq(x'-x)} e^{-ik(x'^0 - x^0)} G_+^{ij}(q, \mathbf{K}) \tag{32}$$

where $k = (q^2 + \mathbf{K}^2)^{1/2}$.

## 3. SOLUTION OF THE GREEN FUNCTIONS

The boundary condition in (13) gives, from (15), (20), (26), and (32),

$$G_{+}^{aj} = \left(\delta^{aj} - \frac{k^a k^j}{k^2}\right) + G_{-}^{aj}, \qquad a = 1, 3 \tag{33}$$

$\mathbf{k} = (q, \mathbf{K})$, $k^1 = q$, $k^2 = K^2$, $k^3 = K^3$; and the boundary condition in (12) gives

$$G_{+}^{2j} = 0 = \left(\delta^{2j} - \frac{k^2 k^j}{k^2}\right) + G_{-}^{2j} \tag{34}$$

The latter, in particular, implies

$$G_{-}^{2j} = -\left(\delta^{2j} - \frac{k^2 k^j}{k^2}\right) \tag{35}$$

The transversality conditions in (19) give, respectively,

$$-qG_{-}^{1j} + K^2 G_{-}^{2j} + K^3 G_{-}^{3j} = 0 \tag{36}$$

$$qG_{-}^{1j} + K^2 G_{-}^{2j} + K^3 G_{-}^{3j} = 0 \tag{37}$$

which, in particular, imply that

$$G_{-}^{1j} = 0 \tag{38}$$

On the other hand, the transversality conditions in (31),

$$k^i G_{+}^{ij} = k^a G_{+}^{aj} = 0 \tag{39}$$

where we have used the fact that $G_{+}^{2j} = 0$ in (34), imply, upon multiplying (33) by $k^a$, $a = 1, 3$, that

$$k^a\left(\delta^{aj} - \frac{k^a k^j}{k^2}\right) + K^3 G_{-}^{3j} = 0 \tag{40}$$

The latter gives

$$G_{-}^{3j} = -\frac{k^a}{K^3}\left(\delta^{aj} - \frac{k^a k^j}{k^2}\right) = \frac{K^2}{K^3}\left(\delta^{2j} - \frac{K^2 k^j}{k^2}\right) \tag{41}$$

Finally, (33) gives, from (38) and (41),

$$G_{+}^{1j} = \left(\delta^{1j} - \frac{qk^j}{k^2}\right) \tag{42}$$

$$G_{+}^{3j} = \delta^{3j} + \frac{K^2}{K^3}\delta^{2j} - \frac{k^j}{k^2}\frac{(K^2)^2 + (K^3)^2}{K^3} \tag{43}$$

The so-called scattering part $G_-^{ij}(x', x)$ of the photon Green function for $x' < 0$ is then defined in (26) with $G_-^{ij}$ given in (38), (35), and (41) for $i = 1$, 2, 3, respectively. This expression is understood to be sandwiched as in $J_2^i(x')D_-^{ij}(x', x)J_1^j(x)$. Similarly, the scattering part $G_+^{ij}(x', x)$ of the photon Green function for $x' > 0$ is defined in (32) with $G_+^{ij}$ given in (42), (34), (43) for $i = 1$, 2, 3, respectively. This expression will be sandwiched as in $J_3^i(x')D_+^{ij}(x', x)J_1^j(x)$.

## 4. TRANSITION AMPLITUDES

The transition amplitude for the process where the photon goes through the polarizer will be extracted from the expression $J_3^iG_+^{ij}J_1^j$ appearing as part of the exponent of the vacuum-to-vacuum transition amplitude in (10). That is, using (32), (42), (34), and (43), it is extracted from

$$\int \frac{d^2\mathbf{K}}{(2\pi)^2} \int \frac{dq}{2\pi} \frac{i}{2k} J_3^{*i}(q, \mathbf{K})G_+^{ij}(q, \mathbf{K})iJ_1^j(q, \mathbf{K}) \tag{44}$$

Without loss of generality, we consider a photon initially with momentum

$$\mathbf{k} = (q, 0, K^3) \tag{45}$$

emitted from $\mathbf{J}_1$, and introduce two mutually orthonormal polarization vectors $\mathbf{e}_1(\theta)$, $\mathbf{e}_2(\theta)$, which in turn are perpendicular to $\mathbf{k}$, with $\mathbf{e}_1(\theta)$ making an angle $\theta$ with the $z$ axis. Thus we may choose

$$\mathbf{e}_1(\theta) = \left(-\frac{K^3}{k}\cos\theta, \sin\theta, \frac{q}{k}\cos\theta\right) \tag{46}$$

where we have used the fact that $|\mathbf{k}|^2 = (K^3)^2 + q^2$. The polarization vector $\mathbf{e}_2(\theta)$ may be written as

$$\mathbf{e}_2(\theta) = \left(-\frac{K^3}{k}\sin\theta, -\cos\theta, \frac{q}{k}\sin\theta\right) \tag{47}$$

We have the completeness relation ($i, j = 1, 2, 3$)

$$e_\lambda^i e_\lambda^j + \frac{k^i k^j}{|\mathbf{k}|^2} = \delta^{ij} \tag{48}$$

with a sum over $\lambda = 1, 2$ understood. We also introduce polarization vectors to describe a photon detected by $J_3$:

$$\mathbf{e}_1' = \left(-\frac{K^3}{k}, 0, \frac{q}{k}\right) \tag{49}$$

$$\mathbf{e}_2' = (0, 1, 0) \tag{50}$$

satisfying also the completeness relation

$$e'^{i}_{\lambda} e'^{j}_{\lambda} + \frac{k^i k^j}{|\mathbf{k}|^2} = \delta^{ij} \tag{51}$$

Since $k^i G^{ij}_{+} = 0$ and $G^{ij}_{+} k^j = 0$, the integrand in (44) may be rewritten as

$$\frac{i}{2k} J^{l*}_3(q, \mathbf{K}) e'^{l}_{\lambda} e'^{i}_{\lambda} G^{ij}_{+} e^{j}_{\alpha} e^{k}_{\alpha} J^{k}_1(q, \mathbf{K})\, i \tag{52}$$

That is, the amplitude that a photon (emitted by $\mathbf{J}_1$), with momentum $\mathbf{k}$ in (45), and polarization $\mathbf{e}_1(\theta)$ making an angle $\theta$ with the $z$ axis, passes through the polarizer (detected by $\mathbf{J}_3$) and ends up with polarization $\mathbf{e}'_1$ is (Manoukian, 1986b)

$$e'^{i}_{1} G^{ij}_{+} e^{j}_{1}(\theta) = \cos\theta \tag{53}$$

On the other hand, the amplitude that the photon ends up polarized along the $y$ axis is explicitly

$$e'^{i}_{2} G^{ij}_{+} e^{j}_{1}(\theta) = 0 \tag{54}$$

as expected. That is, the probability that the photon passes through the polarizer is, from (53), $\cos^2\theta$, which is the classic result.

Now consider the situation where the initial photon is nonpolarized. In this case we also compute explicitly

$$e'^{i}_{1} G^{ij}_{+} e^{j}_{2}(\theta) = \sin\theta \tag{55}$$

$$e'^{i}_{2} G^{ij}_{+} e^{j}_{2}(\theta) = 0 \tag{56}$$

in addition to (53), (54). Then we have to average over the two probabilities obtained from (53) and (55). That is, the probability that a nonpolarized photon passes through the polarizer is

$$\tfrac{1}{2}(\cos^2\theta + \sin^2\theta) = \tfrac{1}{2} \tag{57}$$

The transition amplitude for the process where a photon does *not* pass through the polarizer is extracted from $J^{i*}_2 G^{ij}_{-} J^{j}_1$. That is, it is obtained from

$$\int \frac{d^2\mathbf{K}}{(2\pi)^2} \int \frac{dq}{2\pi} \frac{1}{2k}\, iJ^{i*}_2(-q, \mathbf{K}) G^{ij}_{-} iJ^{j}_1(q, \mathbf{K}) \tag{58}$$

The initial momentum of a photon is taken as in (45) with $K^2 = 0$. From (58) the final momentum of a nontransmitted (reflected!) photon (detected by $\mathbf{J}_2$) is

$$\mathbf{k}'' = (-q, 0, K^3) \tag{59}$$

We introduce polarization vectors to describe a photon detected by $\mathbf{J}_2$:

$$\mathbf{e}_1'' = \left(\frac{K^3}{k}, 0, \frac{q}{k}\right) \tag{60}$$

$$\mathbf{e}_2'' = (0, 1, 0) \tag{61}$$

satisfying the completeness relation

$$e_\lambda''^i e_\lambda''^j + \frac{k''^i k''^j}{|\mathbf{k}|^2} = \delta^{ij} \tag{62}$$

We note that $k''^i G_\perp^{ij} = 0$. The amplitude that a photon (emitted by $\mathbf{J}_1$), with momentum $\mathbf{k}$ in (45), polarization $\mathbf{e}_1(\theta)$ in (46) making an angle $\theta$ with the $z$ axis, does *not* pass the polarizer (reflected!) and ends up with polarization $\mathbf{e}_1''$ is

$$e_1''^i G_\perp^{ij} e_1^j(\theta) = 0 \tag{63}$$

On the other hand, the amplitude for the photon not being transmitted through the polarizer and being polarized along the $y$ axis is explicitly

$$e_2''^i G_\perp^{ij} e_1^j(\theta) = -\sin\theta \tag{64}$$

That is, the nontransmitted (reflected) photon must be polarized along the $y$ axis! The probability that the photon does not pass through the polarizer is, from (64), the classic result $\sin^2\theta$.

Finally, suppose that the initial photon is nonpolarized. In addition to (63), (64) we also compute

$$e_1''^i G_\perp^{ij} e_2^j(\theta) = 0 \tag{65}$$

$$e_2''^i G_\perp^{ij} e_2^j(\theta) = -\cos\theta \tag{66}$$

That is, the probability that a nonpolarized photon does *not* pass through the polarizer (reflected) is, from (64), (65),

$$\tfrac{1}{2}(\sin^2\theta + \cos^2\theta) = \tfrac{1}{2} \tag{67}$$

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